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Jackson and Bernstein-Type Inequalities for Families of Commutative Operators in Banach Spaces

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TO JOSEPH L. WALSH ON HIS 75TH BIRTHDAY, IN ADMIRATION

In some recent papers the authors [7–9] have treated sequences $\{V_n\,; n = 0, 1, 2,...\}$ of (commutative) operators $V_n \in \mathscr{E}(X)$ (= Banach algebra of endomorphisms of the Banach space X) satisfying Jackson and Bernstein-type inequalities. Their results include "direct" and the corresponding "inverse" approximation theorems, theorems of Zamansky-type for such operators, as well as theorems of "reduction" type. Within a certain framework it was shown that the assertions of these four types of theorems are equivalent to each other.

The purpose of this paper is fivefold. First, the sequences depending upon the discrete parameter n, $n \to \infty$, are replaced by the family $\mathscr{U} = \{U(t) : 0 < t \leqslant 1\}$ of operators in $\mathscr{E}(X)$ depending upon the continuous parameter t, $t \to 0+$. While this is a minor modification (and includes the discrete case) the major one consists in broadening the notion of order of approximation by using the general concept of a *function norm* Φ. To be specific, with the notation $\Phi_{\theta,\infty}[\varphi(t)] = \sup_{t\in(0,1]} t^{-\theta}\varphi(t)$, φ being any nonnegative measurable function on (0, 1], the (classical) approximation assertion

$$\| U(t)f - f\| = O(t^{\theta}) \qquad (t \in (0, 1], t \to 0+)$$

may be restated as $\Phi_{\theta,\infty}[\| U(t)f - f\|_X] < \infty$. More generally, the approximation order $O(t^{\theta})$ is to be replaced by $O(\Omega(t))$, $\Omega(t)$ being a monotone increasing function of t on (0, 1] (de la Vallée Poussin [20, Chap. 4] seems to be the first to use such Ω's in approximation theory; see also the account in Timan [19]).

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More generally, one may define†

$$\Phi_{\Omega,q}[\varphi(t)] = \begin{cases} \sup_{t\in(0,1]} \Omega^{-1}(t)\,\varphi(t), & q = \infty \\ \left\{\int_0^1 [\Omega^{-1}(t)\,\varphi(t)]^q\, t^{-1}\, dt\right\}^{1/q}, & 1 \leqslant q < \infty, \end{cases}$$

the case $q = \infty$ giving the approximation $O[\Omega(t)]$. This would give approximation in the setting of the theory of K-intermediate spaces (see the treatment in the case $\Omega(t) = t^\theta$ in [4]). In their most general form, function seminorms Φ will be introduced as functionals satisfying a suitable system of axioms, and fitted into the framework of approximation as above. An approach via functional norms was already followed in interpolation-space theory by Gagliardo [11] and Peetre [17, 18] (see the discussion in Butzer–Berens [4, pp. 213–215]). In this paper, a somewhat more general system of axioms will be set up. This will enable us to present the basic structure and fundamental theorems of linear approximation processes in a systematic and axiomatic fashion.

Third, Jackson and Bernstein-type inequalities will be intensively investigated. In the light of the foregoing, it will be postulated that the family $\mathscr{U}$ satisfies such inequalities of *order* $y(t)$ on X with respect to a second Banach space $Y \subset X$, "$\subset$" in the sense of continuous embedding. In previous papers by various authors [2, 7, 9, 17] such inequalities were considered only in the particular case $y(t) = t^\alpha$, $\alpha > 0$. Given a Jackson and Bernstein-type inequality of order $y(t)$ on X with respect to Y, necessary and sufficient conditions will be established in order that there exist such inequalities of "intermediate" order $z(t)$ on X with respect to certain spaces Z, which are "intermediate" between X and Y. (In the particular instance that $y(t) = t^\alpha$, $z(t) = t^\beta$, this means that $0 < \alpha < \beta$). In most of the applications this boils down to the fact that only Jackson and Bernstein-type inequalities of "highest" possible (or saturation) order need be verified.

Fourth, it will be shown that Jackson and Bernstein-type inequalities for the pairs X, Y and X, Z, with respective orders $y(t)$ and $z(t)$, imply such inequalities for the pair Z, Y with "reduced" order $y(t)(z(t))^{-1}$, provided that $y(t)$ is "better" than $z(t)$ (which implies that $Z \subset Y$). This will enable us to establish the main result (Theorem 6) of this paper, an equivalence theorem on the order of approximation of $U(t)$ to the identity I in the setting of the function seminorm Φ. Whereas this theorem in its preliminary version (Theorem 1) is only concerned with the equivalence of "direct," "inverse," and Zamansky-type assertions, the theorem in its general form includes a

† Note that below $\Omega^{-1}(t)$ (or $(\Omega(t))^{-1}$) always stands for $1/\Omega(t)$, and not for the inverse function.

fourth assertion equivalent to these, namely, one of "reduction" type. By this is meant that approximation is taken in a "stronger" norm, but with a certain loss of the order of approximation. (In the applications, this signifies simultaneous approximation of a function and its derivatives). All in all, the axiomatization presented leads to a clarification and simplification on the one hand and allows a more general theory on the other. The presentation is self-contained, the proofs being carried out in detail.

Fifth, it will be seen that the theory is built up in such a way that it contains the corresponding investigations (see [4]) for holomorphic semigroups of operators $\{T(t) : 0 < t < \infty\}$ of class (C_0) in $\mathscr{E}(X)$, as well as for the family of resolvent operators $\{\lambda R(\lambda; A) : 0 < \lambda < \infty\}$, A being the infinitesimal generator of the semigroup. Moreover, it also includes a large variety of applications to various summation processess of Fourier series, the Riesz means of the Fourier-inversion integral receiving special attention. At the same time this paper provides complete proofs of results announced in Butzer–Scherer [10].

1. Preliminaries

We begin with some basic definitions.

Definition 1. Given a Banach space X, we denote by $\mathscr{M}(X)$ the class of all X-valued functions on (0, 1] which are strongly measurable. In particular, if X is the set R_1^+ of all nonnegative reals, we write $\mathscr{M}(R_1^+) = \mathscr{M}^+$.

Definition 2. A *function*[1] *seminorm* Φ is a functional Φ defined on $\mathscr{M}^+$ which is nontrivial (i.e., there exists a nonnull $\psi \in \mathscr{M}^+$ such that $\Phi(\psi) < \infty$), and satisfies for each $\Phi \in \mathscr{M}^+$:

$$\Phi[\alpha\varphi] = \alpha\Phi(\varphi) \qquad (\alpha \geqslant 0), \tag{1.1}$$

$$\varphi(t) \leqslant \sum_{k=0}^{\infty} \varphi_k(t) \text{ a.e.} \Rightarrow \Phi(\varphi) \leqslant \sum_{k=0}^{\infty} \Phi[\varphi_k], \tag{1.2}$$

$$\Phi[\varphi(t)] < \infty \Rightarrow \varphi(t) < \infty \text{ a.e.} \quad (\varphi \in \mathscr{M}^+). \tag{1.3}$$

In the following, we shall study properties of function seminorms which are important for the theory in the subsequent sections. They are more general than those in [12] and are partly related to well-known inequalities of Hardy.

Definition 3. A function seminorm Φ is *regular* if, for each $\varphi \in \mathscr{M}^+$,

$$\Phi[\varphi(t/2)] \leqslant C_\Phi \Phi[\varphi(t)] \tag{1.4}$$

where C_Φ is a positive constant.

[1] Compare Gagliardo [11], Peetre [17] and Goulaouic [12] for the notion of a function-norm.

DEFINITION 4. Φ is *upper-bounded* by $z(t)$ ($z(t) \in \mathscr{M}^+$), if for some constant $A(\Phi, z) > 0$,

$$\Phi[z(t)\, \bar{\varphi}(t)] \leqslant A(\Phi, z)\, \Phi[z(t)\, \varphi(t)] \qquad (\varphi \in \mathscr{M}^+), \tag{1.5}$$

where

$$\bar{\varphi}(t) = \int_0^t \varphi(u)\, u^{-1}\, du.$$

Φ is *lower-bounded* by $y(t) \in \mathscr{M}^+$, if

$$\Phi[y(t)\, \underline{\varphi}(t)] \leqslant B(\Phi, y)\, \Phi[y(t)\, \varphi(t)] \qquad (\varphi \in \mathscr{M}^+), \tag{1.6}$$

for a constant $B(\Phi, y) > 0$, where

$$\underline{\varphi}(t) = \int_t^1 \varphi(u)\, u^{-1}\, du.$$

LEMMA 1.[2] *Let Φ be a function seminorm.*

(a) *If Φ is upper-bounded by $z_1(t) \in \mathscr{M}^+$, it is also upper-bounded by every $z_2(t) \in \mathscr{M}^+$ such that $z_2(t)/z_1(t)$ is nonincreasing; in particular it is upper-bounded by any constant if $z_1(t)$ is nondecreasing.*

(b) *If Φ is lower-bounded by $y_1(t) \in \mathscr{M}^+$, it is also lower-bounded by every $y_2(t) \in \mathscr{M}^+$ such that $y_2(t)/y_1(t)$ is nondecreasing.*

(c) *If Φ is upper-bounded by a constant and lower-bounded by a nondecreasing bounded $y(t) \in \mathscr{M}^+$, then $\Phi[y(t)] < \infty$.*

Proof. Part (a) follows by (1.2) and (1.6) since

$$\begin{aligned}
\Phi[z_2(t)\, \bar{\varphi}(t)] &= \Phi\left[z_1(t)\{z_2(t)/z_1(t)\} \int_0^t \varphi(u)\, u^{-1}\, du\right] \\
&\leqslant \Phi\left[z_1(t) \int_0^t \{z_2(u)/z_1(u)\}\, \varphi(u)\, u^{-1}\, du\right] \\
&\leqslant A(\Phi, z_1)\, \Phi[z_2(t)\, \varphi(t)].
\end{aligned}$$

Taking $z_2(t) = \textit{const.}$, we see that Φ is upper-bounded by any constant provided $z_1(t)$ is nondecreasing.

In a similar manner, assertion (b) follows in view of

$$\begin{aligned}
\Phi[y_2(t)\, \underline{\varphi}(t)] &\leqslant \Phi[y_1(t) \int_t^1 \{y_2(u)/y_1(u)\}\, \varphi(u)\, u^{-1}\, du] \\
&\leqslant B(\Phi, y_1)\, \Phi[y_2(t)\, \varphi(t)].
\end{aligned}$$

[2] Here and in the following, the monotonicity could be weakened to quasimonotonicity in the sense of Bernstein [3].

Finally, let ψ be a nonnull function in $\mathscr{M}^+$ with $\Phi[\psi] < \infty$. Then $\Phi[\bar{\psi}(t)] < \Phi[\psi] < \infty$ since Φ is upper-bounded by a constant. By (1.3) it follows that

$$\int_0^{t_0} \psi(u)\, u^{-1}\, du < \infty$$

for some $t_0 \in (0, 1]$. Hence, for the function $\psi^*(u) = \psi(u)$ for $0 < u \leqslant t_0$, $= 0$ for $t_0 < u \leqslant 1$, one has

$$0 < \int_0^1 \psi^*(u)\, du < \infty$$

and $\Phi[\psi^*] < \infty$, by (1.2). Now, applying (1.5) for $z(t) = 1$ and (1.6), we obtain, by (1.2),

$$\begin{aligned}
\Phi[y(t)] &\leqslant \left[\int_0^1 \psi(t)\, dt\right]^{-1} \left\{ y(1)\Phi\left[\int_0^t \psi(u)\, uu^{-1}\, du\right] + \Phi\left[y(t)\int_t^1 \psi(u)\, uu^{-1}\, du\right]\right\} \\
&\leqslant \left[\int_0^1 \psi(t)\, dt\right]^{-1} \{y(1)\, A(\Phi, 1)\, \Phi[\psi(t)t] + B(\Phi, y)\, \Phi[y(t)\, \psi(t)\, t]\} \\
&\leqslant \left[\int_0^1 \psi(t)\, dt\right]^{-1} [A(\Phi, 1) + B(\Phi, y)]\, y(1)\, \Phi[\bar{\psi}] < \infty.
\end{aligned}$$

Lemma 1 already justifies in some sense the terminology of lower- and upper-boundedness of Φ. This becomes still more apparent when one considers the most important examples of Φ, namely,

$$\begin{aligned}
\Phi_{\Omega,q}[\varphi] &= \left\{\int_0^1 [\Omega^{-1}(t)\, \varphi(t)]^q\, t^{-1}\, dt\right\}^{1/q} \qquad (1 \leqslant q < \infty), \\
\Phi_{\Omega,\infty}[\varphi] &= \sup_{t\in(0,1]} \Omega^{-1}(t)\, \varphi(t) \qquad (q = \infty),
\end{aligned}$$

where $\Omega(t)$ is a positive nondecreasing function of $\mathscr{M}^+$. In the case $\Omega(t) = t^\theta$, $\theta > 0$, we shall write $\Phi_{t^\theta,q} = \Phi_{\theta,q}$. It is easily verified that $\Phi_{\Omega,q}$ is a function seminorm, since it satisfies conditions (1.1)–(1.3) and is nontrivial in view of

$$\Phi_{\Omega,q}[\Omega(t)t] = \Phi_{1,q}[t] = \left\{\int_0^1 t^{q-1}\, dt\right\}^{1/q} < \infty.$$

It is, moreover, regular[3] since

$$\Phi_{\Omega,q}[\varphi(t/2)] \leqslant \left\{\int_0^1 [\Omega^{-1}(t/2)\, \varphi(t/2)]^q\, t^{-1}\, dt\right\}^{1/q} \leqslant \Phi_{\Omega,q}[\varphi(t)].$$

[3] We remark that in order to show the regularity of Φ it is sufficient to assume $\Omega(t/2) \leqslant const.\ \Omega(t)$ instead of assuming $\Omega(t)$ to be nondecreasing.

We shall now investigate how lower- and upper-boundedness may be expressed in terms of $\Omega(t)$.

LEMMA 2. *Let $\Omega(t)$, $y(t)$, and $z(t)$ be positive functions in $\mathscr{M}^+$ and let $\Omega(t)$ be nondecreasing.*

(a) *The function seminorm $\Phi_{\Omega,\infty}$ is upper-bounded by $z(t)$ if and only if*

$$\int_0^t z^{-1}(u)\,\Omega(u)\,u^{-1}\,du = O[z^{-1}(t)\,\Omega(t)], \tag{1.7}$$

and is lower-bounded by $y(t)$ if and only if

$$\int_t^1 y^{-1}(u)\,\Omega(u)\,u^{-1}\,du = O[y^{-1}(t)\,\Omega(t)]. \tag{1.8}$$

(b) *The function seminorm $\Phi_{\Omega,1}$ is upper-bounded by $z(t)$ if and only if*

$$\int_t^1 z(u)\,\Omega^{-1}(u)\,u^{-1}\,du = O[z(t)\,\Omega^{-1}(t)], \tag{1.9}$$

and is lower-bounded by $y(t)$ if and only if

$$\int_0^t y(u)\,\Omega^{-1}(u)\,u^{-1}\,du = O[y(t)\,\Omega^{-1}(t)]. \tag{1.10}$$

(c) *The function seminorm $\Phi_{\Omega,q}$, $1 < q < \infty$, is upper-bounded by $z(t)$ if both* (1.7) *and* (1.9) *are valid, and is lower-bounded by $y(t)$ if both* (1.8) *and* (1.10) *are valid.*

Proof. Setting $g(u) = \varphi(u)\,z(u)\,\Omega^{-1}(u)\,u^{-1}$,

$$V[g(u); t] = t^{-1}z(t)\,\Omega^{-1}(t)\int_0^t g(u)\,z^{-1}(u)\,\Omega(u)\,du$$

and

$$h(u) = \varphi(u)\,y(u)\,\Omega^{-1}(u)\,u^{-1},\ W[h(u); t] = t^{-1}y(t)\,\Omega^{-1}(t)\int_t^1 h(u)\,y^{-1}(u)\,\Omega(u)\,du,$$

we can rewrite conditions (1.5) and (1.6), in case $1 \leqslant q < \infty$, in the form

$$\| V[g(u); t] \|_{L_q[0,1]} \leqslant A(\Omega, q, z) \| g \|_{L_q[0,1]},$$

$$\| W[h(u); t] \|_{L_q[0,1]} \leqslant B(\Omega, q, y) \| h \|_{L_q[0,1]},$$

respectively, where $\|\cdot\|_{L_q[0,1]}$ denotes the usual norm of the space of all functions in $\mathscr{M}^+$ whose qth power is Lebesgue-integrable. Since one can assume that the right sides of these inequalities are finite, conditions (1.5) and (1.6) state that V and W, regarded as linear operators on $L_q[0, 1]$ into itself, are bounded. In case $q = 1$, this is equivalent to the assertions

$$\| V\|_{[L_1,L_1]} = \sup_{u\in(0,1]} z^{-1}(u)\,\Omega(u) \int_u^1 z(t)\,\Omega^{-1}(t)\,t^{-1}\,dt < \infty,$$

$$\| W\|_{[L_1,L_1]} = \sup_{u\in(0,1]} y^{-1}(u)\,\Omega(u) \int_0^u y(t)\,\Omega^{-1}(t)\,t^{-1}\,dt < \infty,$$

which are in turn equivalent to (1.9) and (1.10), respectively. A similar argument, in case $q = \infty$, proves part (a). Then the general case $1 < q < \infty$ in part (c) follows from the cases $q = 1$ and $q = \infty$ by the well-known interpolation theorem of Riesz-Thorin applied to the operators V and W.

We remark that for $z(t) = 1$ and $y(t) = t^k$, conditions (1.7) and (1.8) are precisely those on generalized moduli of continuity used by Bari–Stečkin [1] in order to establish some approximation theoretic equivalence theorems. Furthermore, part (c) contains as a particular case well-known inequalities of Hardy (see Hardy–Littlewood–Pólya [14, pp. 245–246]), in which $z(u)\Omega^{-1}(u)$ and $y(u)\,\Omega^{-1}(u)$ are of the form $t^{-\alpha}$ and t^{β}, respectively, with $\alpha, \beta > 0$. In this case one easily verifies (1.7)–(1.10). Specializing further, with $\Omega(t) = t^{\theta}$, $z(t) = t^k$ and $y(t) = t^l$, $\theta > 0$ and k, l being nonnegative integers, conditions (1.7), (1.9) are equivalent to $\theta > k$, and (1.8), (1.10) to $\theta < l$.

Function seminorms will be now employed to construct Banach subspaces of a given Banach space X.

DEFINITION 5. Let X be a Banach space. We denote by $\mathscr{N}^+(X)$ the class of all functionals on the product space $(0, 1] \times X$ into R_1^+ whose compositions with the projections on X and $(0, 1]$ are continuous seminorms on X and belong to $\mathscr{M}^+$, respectively.

DEFINITION 6. We denote by $X(\Phi; M)$ the subspace of all elements $f \in X$ such that $\Phi[M(t,f)] < \infty$, where Φ is a function seminorm and $M(t,f)$ a functional belonging to $\mathscr{N}^+(X)$.

DEFINITION 7. A subspace Y of the Banach space X is said to be a *normal* Banach subspace of X if there is a seminorm $|\cdot|_Y$ defined on Y such that Y is a Banach space with respect to the norm $\|\cdot\|_Y = \|\cdot\|_X + |\cdot|_Y$.

LEMMA 3. *$\Phi[M(t,f)]$ is a seminorm, and the space $X(\Phi, M)$ is a normal Banach subspace of X with respect to the norm $\|f\|_{\Phi,M} \equiv \|f\|_X + \Phi[M(t,f)]$.*

Proof. First, it is obvious that $\|f\|_{\Phi,M} = 0$ if and only if $f = 0$. Second, one has by (1.1) and (1.2),

$$\Phi[M(t, \alpha_1 f_1 + \alpha_2 f_2)] \leq \Phi[|\alpha_1| M(t, f_1) + |\alpha_2| M(t, f_2)]$$
$$\leq |\alpha_1| \Phi[M(t, f_1)] + |\alpha_2| \Phi[M(t, f_2)].$$

It remains to show that $X(\Phi; M)$ is a Banach space with respect to the indicated norm. This is equivalent to the fact that for every sequence $\{f_n\}_1^\infty$, $f_n \in X(\Phi; M)$, with $\sum_1^\infty \|f_n\|_{\Phi,M} < \infty$, we have $\sum_1^\infty f_n \in X(\Phi; M)$. But for such a sequence also $\sum_1^\infty \|f_n\|_X$ is convergent, and hence $\sum_1^\infty f_n = f \in X$ since X is complete. Furthermore, the property (1.2) of Φ together with the continuity of the seminorm $M(t, f)$ yield

$$\|f\|_{\Phi,M} = \Big\| \sum_{n=1}^{\infty} f_n \Big\|_X + \Phi\left[M\left(t, \sum_{n=1}^{\infty} f_n\right)\right]$$

$$\leq \sum_{n=1}^{\infty} \{\|f_n\|_X + \Phi[M(t, f_n)]\} < \infty,$$

so that $f \in X(\Phi; M)$.

A special instance of a functional belonging to $\mathcal{N}^+(X)$ is the K-functional of Peetre [16] (see [4, p. 166]). If Y is any normal Banach subspace of X, it is given (in a modified form) by

$$K(t, f; X, Y) \equiv \inf_{g \in Y} (\|f - g\|_X + t\,|g|_Y) \qquad (f \in X;\ 0 < t < \infty).$$

LEMMA 4. *The K-functional $K(t, f; X, Y)$ has the following properties*:

$$K(t_1, f; X, Y) \leq \max(1, t_1 t_2^{-1})\, K(t_2, f; X, Y) \qquad (t_1, t_2 \in (0, \infty)),$$

$$K(t, f; X, Y) \leq \|f\|_X \qquad (f \in X,\ t \in (0, \infty)),$$

$$K(t, f; X, Y) \leq t\,|f|_Y \qquad (f \in Y,\ t \in (0, \infty)).$$

We omit the proof of these relations which are almost obvious. They show that the K-functional is monotone increasing as a function of t, and for a fixed t is a continuous seminorm on X and thus of class $\mathcal{N}^+(X)$. Therefore, in view of Lemma 3, the spaces

$$X(\Phi^{(y)}; K) \equiv (X, Y; \Phi^{(y)}) = \{f \in X : \Phi[K(y(t), f; X, Y)] < \infty\},$$

where $\Phi^{(y)}[\varphi(t)] = \Phi[\varphi(y(t))]$ and $y(t)$ is monotone increasing with

$0 < y(t) \leqslant 1$,[4] are normal Banach subspaces of X. If Φ is regular and lower-bounded by $y(t)$, the inequality

$$\Phi[K(y(t), f; X, Y)] \leqslant \Phi[y(t)] \, |f|_Y \qquad (f \in Y),$$

is valid. We can abbreviate this in the form

$$Y < (X, Y; \Phi^{(y)}) \subset X, \tag{1.11}$$

where $X_1 \subset X_2$, for Banach spaces X_1, X_2, means that $\|f\|_{X_2} \leqslant M \|f\|_{X_1}$ for every $f \in X_1$, $M > 0$ being a constant, and $Y_1 < Y_2$, for normal Banach subspaces Y_1, Y_2 of X, means that $|f|_{Y_2} \leqslant M' \, |f|_{Y_1}$ for every $f \in Y_1$, $M' > 0$ being a constant. In view of (1.11), the spaces $(X, Y; \Phi^{(y)})$ are called K-intermediate spaces.

A representative example of such spaces is obtained when Y is equal to $D(A^r)$, i.e., to the domain of the rth power of the infinitesimal generator A of a one-parameter semigroup $\mathscr{T} = \{T(t) : 0 \leqslant t < \infty\}$ of operators of class (C_0), of a family of operators of $\mathscr{E}(X)$, the Banach algebra of endomorphisms of X, which satisfy for each $f \in X$,

$$\begin{gathered} T(0) = I, \qquad T(t_1 + t_2) = T(t_1)\, T(t_2) \qquad (t_1, t_2 > 0), \\ \lim_{t \to 0+} \| T(t) f - f \|_X = 0. \end{gathered} \tag{1.12}$$

In this case we set $|f|_Y = \| A^r f \|_X$ and $\|f\|_Y = \|f\|_X + \| A^r f \|_X$ for $f \in D(A^r)$, so that $D(A^r)$ is a normal Banach subspace of X. Now there exists a fundamental relation between $K(t^r, f; X, D(A^r))$ and the rth modulus of continuity of the semigroup defined by

$$\omega_r(t, f; \mathscr{T}) \equiv \sup_{|h| \leqslant t} \| [T(h) - I]^r f \|_X .$$

LEMMA 5. *For every $f \in X$ and $t \in (0, 1]$,*

$$\begin{aligned} \omega_r(t, f; \mathscr{T}) &\leqslant 2^r (M_{\mathscr{T}} + 1)^r \, K(t^r, f; X, D(A^r)) \\ &\leqslant (4r)^r (M_{\mathscr{T}} + 1)^r \, \omega_r(t, f; \mathscr{T}), \end{aligned} \tag{1.13}$$

where $M_{\mathscr{T}} = \sup_{t \in (0,1]} \| T(t) \| < \infty$.

Proof. The left side of the inequality (1.13) was established in Butzer–Berens [4]. Since an explicit proof of the right side is to be found in [4] only

[4] Concerning more general conditions which assure that $\varphi(y(t))$ belongs to $\mathscr{M}^+$ for every $\varphi \in \mathscr{M}^+$, see Halmos [13, p. 81].

for $r = 1, 2$ and in a weaker form, we shall prove it here. First, we note the relation

$$[T(t) - I]^r f = A^r \int_0^t du_1 \cdots \int_0^t T\left(\sum_{i=1}^r u_i\right) f\, du_r$$

which is easily proved by induction. Therefore

$$g_{r,l} \equiv (r/t)^r \int_0^{t/r} du_1 \cdots \int_0^{t/r} T\left(l \sum_{i=1}^r u_i\right) f\, du_r$$

is an element of $D(A^r)$, for $l = 1, 2, \ldots$, satisfying

$$A^r g_{r,l} = (r/(lt))^r [T(lt/r) - I]^r f.$$

Next, we set $u = \sum_{i=1}^r u_i$ and choose

$$g = -(-1)^r \sum_{l=1}^r (-1)^{r-l} \binom{r}{l} g_{r,l}$$

in the representation $f = (f - g) + g$. Then

$$\begin{aligned}
K(t^r, f; X, D(A^r)) &\leqslant \|f - g\|_X + t^r \|A^r g\|_X \\
&= \left\| \sum_{l=0}^r (-1)^{r-1} \binom{r}{l} g_{r,l} \right\|_X + t^r \|A^r g\|_X \\
&\leqslant (r/t)^r \int_0^{t/r} du_1 \cdots \int_0^{t/r} \|[T(u) - I]^r f\|_X\, du_r \\
&\qquad + \sum_{l=1}^r \binom{r}{l} (r/l)^r \|[T(lt/r) - I]^r f\|_X \\
&\leqslant \omega_r(t, f; \mathscr{T}) + \sum_{l=1}^r \binom{r}{l} (r/l)^r \omega_r(t, f; \mathscr{T}) \\
&\leqslant (2r)^r \omega_r(t, f; \mathscr{T}),
\end{aligned}$$

which was to be proved.

As a consequence of (1.13) we obtain the following characterization of K-intermediate spaces of X and $D(A^r)$, namely

$$(X, D(A^r); \Phi^{(r)}) = \{f \in X : \Phi[\omega_r(t, f; \mathscr{T})] < \infty\}, \tag{1.14}$$

where $\Phi^{(r)}[\varphi(t)] = \Phi[\varphi(t^r)]$. In the particular instance when the semigroup

$\{T(t) : 0 \leqslant t < \infty\}$ is equi-bounded, i.e., $\| T(t)\| \leqslant M_{\mathscr{T}}$ for all $t \in (0, \infty)$, and $\Phi = \Phi_{\theta,q}$, this means that $(\theta > 0)$,

$$(X, D(A^r); \Phi_{\theta,q}^{(r)}) = \left\{f \in X: \left(\int_0^\infty [t^{-\theta}\omega_r(t, f; \mathscr{T})]^q \, t^{-1}\, dt\right)^{1/q} < \infty\right\},$$

since, in this case,

$$\int_1^\infty [t^{-\theta}\omega_r(t, f; \mathscr{T})]^q \, t^{-1}\, dt \leqslant (M_{\mathscr{T}} + 1)^{rq} \|f\|_X \int_1^\infty t^{-\theta q - 1}\, dt < \infty.$$

The latter spaces are denoted in [4] by $X_{\theta,r;q}$. They are nontrivial subspaces of X containing $D(A^r)$ for $0 \leqslant \theta < r$, $1 \leqslant q < \infty$ and $0 \leqslant \theta \leqslant r$, $q = \infty$, since here $\Phi_{\theta,q}[t^r] < \infty$ (see (1.11)). Furthermore, the following reduction theorem of Lions–Peetre [15] is valid for $0 < k < \theta < r$, $1 \leqslant q \leqslant \infty$ (compare also [4, p. 198]):

$$X_{\theta,r;q} = \{f \in D(A^k) : A^k f \in X_{\theta-k,l;q}\}, \tag{1.15}$$

l being any natural number, $l > \theta - k$.

We conclude this section by mentioning that if $\mathscr{T}$ is the semigroup of translations on the Lebesgue space $L_p(-\infty, \infty)$, $1 < p \leqslant \infty$, the spaces $[L_p(-\infty, \infty)]_{\theta,r;q}$ are known as generalized Lipschitz spaces,

$$[T(h) - I]^r f(x) \equiv \Delta_h{}^r f(x) = \sum_{j=0}^{r} \binom{r}{j} (-1)^{r-j} f(x + jh) \qquad (f \in L_p(-\infty, \infty))$$

being the rth difference of $f \in L_p(-\infty, \infty)$ and $\omega_r(t, f; \mathscr{T})$, the familiar rth modulus of continuity of f.

2. Jackson and Bernstein-Type Inequalities of Fixed Order

In this section we consider approximation processes on a Banach space X which are more general than those given by a (holomorphic) semigroup of operators. They are defined by a family $\mathscr{U} = \{U(t) : t \in [0, 1]\}$ of operators of $\mathscr{E}(X)$ satisfying for every $f \in X$:

$$U(t) f \in \mathscr{M}(X), \qquad \| U(t) f\|_X \leqslant M_{\mathscr{U}} \|f\|_X \qquad (t \in [0, 1]), \tag{2.1}$$

$$U(0) = I, \qquad U(t_1)\, U(t_2) = U(t_2)\, U(t_1) \qquad (t_1, t_2 \in [0, 1]), \tag{2.2}$$

$$\lim_{t\to 0+} \| U(t) f - f\|_X = 0. \tag{2.3}$$

The essential properties are the commutativity in (2.2) and the convergence property (2.3) which allows one to speak of a linear approximation process on X. The uniform boundedness property and the strong measurability in (2.1) are assumed for convenience in the subsequent evaluations. A comparison with (1.12) shows that the strong semigroup property is replaced by the weaker property of commutativity. In this way we cover a wide class of linear approximation processes as the discussion in Section 4 will show.

In the following, we shall be concerned with the approximation behaviour of $U(t)$ in (2.3) in more detail. For this purpose, we shall study the Banach spaces

$$X(\Phi; \mathscr{U}) = \{f \in X : \|f\|_{\Phi;\mathscr{U}} = \|f\|_X + \Phi[\| U(t)f - f\|_X] < \infty\}.$$

Here we use $\| U(t)f - f\|_X$ as a functional of class $\mathscr{N}^+(X)$ (which follows from (2.1)) to construct normal Banach subspaces of X generated by $\mathscr{U}$ in the same way as the K-functional was used above to construct the K-intermediate spaces. In the particular case $\Phi = \Phi_{\Omega,q}$, we write $X(\Phi; \mathscr{U}) = X_{\Omega,q;\mathscr{U}}$. Thus, e.g.,

$$X_{\Omega,\infty;\mathscr{U}} = \{f \in X : \| U(t)f - f\|_X = O[\Omega(t)],\ t \to 0+\}$$

represents the space of all elements $f \in X$ which are approximated by $\mathscr{U}$ with order $\Omega(t)$.

We now assume more specific properties concerning the behavior of $U(t)f$ for a normal Banach subspace Y of X, in the form of Jackson and Bernstein-type inequalities.

Definition 8. Let X and Y be Banach spaces as in Section 1, $Y \subset X$, and let $y(t)$ be a monotone increasing function in $\mathscr{M}^+$ satisfying $0 < y(t) \leqslant y(1) = 1$ and

$$y(t) \leqslant m_y y(t/2) \qquad (t \in (0, 1]). \tag{2.4}$$

We say that $\mathscr{U}$ satisfies a *Jackson-type inequality of order $y(t)$ on X with respect to Y* provided

$$\| U(t)f - f\|_X \leqslant C_Y y(t) \, | f |_Y \qquad (f \in Y) \tag{2.5}$$

for some constant $C_Y > 0$, and a *Bernstein-type inequality of order $y(t)$ on X with respect to Y* provided

$$U(t)f \in \mathscr{M}(Y), \qquad | U(t)f |_Y \leqslant D_Y y^{-1}(t) \, \|f\|_X \qquad (f \in X) \tag{2.6}$$

for some constant $D_Y > 0$.

A discussion concerning existence and applicability of Jackson and Bernstein-type inequalities for concrete examples is given in [10] and also in the following Sections 3 and 4. These inequalities enable us to prove the following approximation-theoretic assertions concerning $\mathscr{U}$.

THEOREM 1. *Let $\mathscr{U}$ be a linear approximation process on X defined by* (2.1)–(2.3), *let $U(t)f \in \mathscr{M}(Y)$ for every $f \in X$, and let Φ be a regular function seminorm in the sense of Definition* 3.

(a) *If $\mathscr{U}$ satisfies the Jackson-type inequality* (2.5) *of order $y(t)$ on X with respect to Y, and Φ is upper-bounded by a constant, i.e.,*

$$\Phi[\bar{\varphi}(t)] \leqslant const.\ \Phi[\varphi(t)] \qquad (\varphi \in \mathscr{M}^+),$$

then $\Phi[y(t) \mid U(t)f \mid_Y] < \infty$ implies $\Phi[\| U(t)f - f \|_X] < \infty$.

(b) *If $\mathscr{U}$ satisfies the Bernstein-type inequality* (2.6) *of order $y(t)$ on X with respect to Y, and Φ is upper-bounded by a constant and lower-bounded by $y(t)$, i.e.,*

$$\Phi[y(t)\, \bar{\varphi}(t)] \leqslant const.\ \Phi[y(t)\, \varphi(t)] \qquad (\varphi \in \mathscr{M}^+),$$

then $\Phi[\| U(t)f - f \|_X] < \infty$ implies $\Phi[y(t) \mid U(t)f \mid_Y] < \infty$.

(c) *Under the assumptions of part* (a) *as well as of part* (b), *the assertions $\Phi[y(t) \mid U(t)f \mid_Y] < \infty$ and $\Phi[\| U(t)f - f \|_X] < \infty$ are equivalent, and furthermore*

$$X(\Phi; \mathscr{U}) = (X, Y; \Phi^{(y)}) \tag{2.7}$$

with equivalent norms.

Proof. We first establish part (c). By a standard device, using (2.1) and (2.5), we have, for arbitrary $f \in X$, $g \in Y$,

$$\begin{aligned} \| U(t)f - f \|_X &\leqslant \| [U(t) - I](f - g) \|_X + \| [U(t) - I] g \|_X \\ &\leqslant M_{\mathscr{U}} \| f - g \|_X + C_Y y(t) \mid g \mid_Y, \end{aligned}$$

and hence

$$\| U(t)f - f \|_X \leqslant \max(M_{\mathscr{U}}, C_Y)\, K(y(t), f; X, Y).$$

On the other hand, choosing, in view of $U(t)f \in Y$, the particular representation $f = [f - U(t)f] + U(t)f$, we obtain, by definition,

$$K(y(t), f; X, Y) \leqslant \| U(t)f - f \|_X + y(t) \mid U(t)f \mid_Y .$$

Now, if we assume that $\Phi[\| U(t)f - f \|_X] < \infty$ and $\Phi[y(t) \mid U(t)f \mid_Y] < \infty$

are equivalent, it follows immediately by properties (1.1) and (1.2) of Φ and the above estimates that these assertions are also equivalent to $\Phi[K(y(t), f; X, Y)] < \infty$. This proves part (c).

It remains to prove the equivalence expressed by parts (a) and (b). First, we use the Jackson-type inequality (2.5) which yields, for every pair $u, v \in (0, 1]$, by the commutativity (2.2),

$$\begin{aligned} \| U(v)f - U(u)f \|_X &\leqslant \|[I - U(u)]\, U(v)f\|_X + \|[U(v) - I]\, U(u)f\|_X \\ &\leqslant C_Y[y(u) \mid U(v)f \mid_Y + y(v) \mid U(u)f \mid_Y]. \end{aligned} \tag{2.8}$$

The counterpart follows by the Bernstein-type inequality (2.6) and (2.2), namely

$$\begin{aligned} | U(v)f - U(u)f |_Y &\leqslant | U(v)[I - U(u)]f |_Y + | U(u)[U(v) - I]f |_Y \\ &\leqslant D_Y[y^{-1}(v) \| U(u)f - f\|_X + y^{-1}(u) \| U(v)f - f \|_X]. \end{aligned} \tag{2.9}$$

The rest of the proof now consists of utilizing these inequalities in a suitable technical manner, where conditions (1.4)–(1.7) and (2.4) upon Φ and $y(t)$ are needed. To this end, we introduce the Bochner integral

$$S(t)f = (2/t) \int_{t/2}^{t} U(v)f\, dv \qquad (f \in X),$$

for $t \in (0, 1]$, which is, by assumption, an element of Y and tends to f in X-norm if $t \to 0$. Then the inequalities

$$\begin{aligned} &\sum_{k=0}^{\infty} \| S(t2^{-k})f - S(t2^{-k-1})f \|_X \\ &\quad = \sum_{k=0}^{\infty} \left\| t^{-1}2^{k+1} \int_{t2^{-k-1}}^{t2^{-k}} [U(v)f - U(v/2)f]\, dv \right\|_X \\ &\quad \leqslant \sum_{k=0}^{\infty} t^{-1}2^{k+1} \int_{t2^{-k-1}}^{t2^{-k}} \| U(v)f - U(v/2)f \|_X\, dv \\ &\quad \leqslant 2 \int_0^t \| U(v)f - U(v/2)f \|_X\, v^{-1}\, dv, \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} | S(t)f |_Y &\leqslant \int_{t/2}^{t} | U(v)f |_Y\, v^{-1}\, dv \\ &= \int_{1/2}^{1} | U(v)f |_Y\, v^{-1}\, dv + \int_t^1 [| U(v/2)f |_Y - | U(v)f |_Y]\, v^{-1}\, dv \\ &\leqslant \int_{1/2}^{1} | U(v)f |_Y\, v^{-1}\, dv + \int_t^1 | U(v/2)f - U(v)f |_Y\, v^{-1}\, dv \end{aligned} \tag{2.11}$$

are valid.

Then, applying (2.8) in (2.10), we obtain, by the properties of $y(t)$ for $f \in X$,

$$\begin{aligned}
&\| U(t)f - f\|_X \\
&\quad \leqslant \| U(t)f - S(t)f\|_X + \sum_{k=0}^{\infty} \| S(t2^{-k})f - S(t2^{-k-1})f\|_X \\
&\quad \leqslant 2 \int_{t/2}^{t} \| U(t)f - U(v)f\|_X \, v^{-1}\, dv + 2 \int_0^t \| U(v)f - U(v/2)f\|_X \, v^{-1}\, dv \\
&\quad \leqslant 2C_Y \int_{t/2}^{t} [y(t) \mid U(v)f\mid_Y + y(v) \mid U(t)f\mid_Y] \, v^{-1}\, dv \\
&\qquad\qquad + 2C_Y \int_0^t [y(v) \mid U(v/2)f\mid_Y + y(v/2) \mid U(v)f\mid_Y] \, v^{-1}\, dv \\
&\quad \leqslant 2C_Y \left(m_y \int_{t/2}^{t} y(v) \mid U(v)f\mid_Y v^{-1}\, dv + y(t) \mid U(t)f\mid_Y\right) \\
&\qquad\qquad + 2C_Y(m_Y + 1) \int_0^t y(v) \mid U(v)f\mid_Y v^{-1}\, dv \\
&\quad \leqslant 2C_Y \left[(2m_y + 1) \int_0^t y(v) \mid U(v)f\mid_Y v^{-1}\, dv + y(t) \mid U(t)f\mid_Y\right].
\end{aligned}$$

Part (a) now follows by the properties (1.1), (1.2) and (1.5) of Φ.

To prove part (b), we apply (2.9) in (2.11) and obtain, by the properties of $y(t)$ for $f \in X$,

$$\begin{aligned}
\mid U(t)f\mid_Y &\leqslant \mid U(t)f - S(t)f\mid_Y + \mid S(t)f\mid_Y \\
&\leqslant 2 \int_{t/2}^{t} \mid U(t)f - U(v)f\mid_Y v^{-1}\, dv + \int_{1/2}^{1} \mid U(v)f\mid_Y v^{-1}\, dv \\
&\qquad\qquad + \int_t^1 [\mid U(v/2)f - U(v)f\mid_Y v^{-1}\, dv \\
&\leqslant 2D_Y \int_{t/2}^{t} [y^{-1}(t) \| U(v)f - f\|_X + y^{-1}(v) \| U(t)f - f\|_X] \, v^{-1}\, dv \\
&\qquad + D_Y \int_{1/2}^{1} \| f\|_X \, y^{-1}(v)\, v^{-1}\, dv + D_Y \int_t^1 [y^{-1}(v) \| U(v/2)f - f\|_X \\
&\qquad\qquad + y^{-1}(v/2) \| U(v)f - f\|_X] \, v^{-1}\, dv \\
&\leqslant 2D_Y \left[\int_{t/2}^{1} y^{-1}(v) \| U(v)f - f\|_X \, v^{-1}\, dv + m_y y^{-1}(t) \| U(t)f - f\|_X\right. \\
&\qquad + D_Y \| f\|_X \, y^{-1}(1/2) \\
&\qquad\quad + D_Y(1 + m_y) \int_{t/2}^{1} y^{-1}(v) \| U(v)f - f\|_X v^{-1}\, dv.
\end{aligned}$$

Then the properties (1.1), (1.2) and (1.4) of Φ as well as the upper-boundedness by a constant, the lower boundedness (1.6) by $y(t)$ and the regularity of Φ yield the implication of part (b), since by the latter condition, $\Phi[y(t)] < \infty$ (see Lemma 1(c)).

We remark that part (b) contains an assertion of Zamansky's type (who first proved assertions of this kind for sequences of trigonometric polynomials) and that part (c) establishes what are called in approximation theory, a "direct" and a "converse" theorem, since it allows to conclude convergence properties of the linear approximation process $\mathcal{U}$ given by $\Phi[\| U(t)f-f\|_X] < \infty$ from the structural properties of the spaces X and Y given through $\Phi[K(y(t), f; X, Y)] < \infty$, and conversely. In the next section we shall enlarge these results by assertions of reduction type which give convergence in stronger norms than that of X.

3. Jackson and Bernstein-type inequalities of different orders

The first problem of this section may be described as follows: Given a normal Banach subspace Z of X, we ask for necessary and sufficient conditions which assure that $\mathcal{U}$ satisfies Jackson and Bernstein-type inequalities on X with respect to Z.

THEOREM 2. *The approximation process $\mathcal{U}$ satisfies a Jackson-type inequality of order $z(t) \in \mathcal{M}^+$ on X with respect to a normal Banach subspace Z of X, i.e.,*

$$\| U(t)f-f\|_X \leqslant C_Z\ z(t)\, |f|_Z \qquad (f \in Z;\ t \in (0, 1]), \tag{3.1}$$

for some constant $C_Z > 0$, if and only if

$$Z < X_{z,\infty;\mathcal{U}}\,. \tag{3.2}$$

Proof. By definition, (3.2) states that

$$\Phi_{z,\infty}[\| U(t)f-f\|_X] = \sup_{t\in(0\ 1]} z^{-1}(t)\, \| U(t)f-f\|_X \leqslant C_Z\, |f|_Z$$

for all $f \in Z$. But this is equivalent to (3.1).

The characterization of Bernstein-type inequalities is more complicated. We assume from here on—even if not explicitly mentioned—that $z(t)$ is monotone increasing with $1 = z(1) > z(t) > 0$ for $t \in (0, 1]$, and that $z(t)$ satisfies condition (2.4) with a corresponding constant $m_z > 0$.

THEOREM 3. *If the approximation process $\mathcal{U}$ satisfies a Bernstein-type* *inequality of order $z(t)$ on X with respect to a normal Banach subspace Z of X, i.e.,*

$$U(t)f \in \mathscr{M}(Z), \qquad | U(t)f |_Z \leqslant D_Z \; z^{-1}(t) \, \| f \|_X \qquad (f \in X;\, t \in (0, 1]) \tag{3.3}$$

for some constant $D_Z > 0$, then

$$X_{z,1;\mathscr{U}} \subset Z. \tag{3.4}$$

Proof. Since $U(t)f \in \mathscr{M}(Z)$, it follows that $S(t)f \in Z$ for every $f \in X$, $t \in (0, 1]$. We then proceed as in (2.10) and (2.9), replacing X and Y by Z, respectively, to deduce

$$\begin{aligned}
\sum_{k=0}^{\infty} | S(t2^{-k})f - S(t2^{-k-1})f |_Z &\leqslant 2 \int_0^t | U(v)f - U(v/2)f |_Z v^{-1} \, dv \\
&\leqslant 2D_Z \int_0^t [z^{-1}(v/2) \, \| U(v)f - f \|_X + z^{-1}(v) \, \| U(v/2) f - f \|_X] \, v^{-1} \, dv \\
&\leqslant 2D_Z(1 + m_z) \int_0^t z^{-1}(v) \, \| U(v)f - f \|_X \, v^{-1} \, dv,
\end{aligned}$$

and in a similar manner

$$\begin{aligned}
\sum_{k=0}^{\infty} \| S(t2^{-k})f - S(t2^{-k-1})f \|_X &\leqslant 2 \int_0^t \| U(v)f - U(v/2)f \|_X \, v^{-1} \, dv \\
&\leqslant 4 \int_0^t \| U(v)f - f \|_X \, v^{-1} \, dv \leqslant 4 \int_0^t z^{-1}(v) \, \| U(v)f - f \|_X \, v^{-1} \, dv.
\end{aligned}$$

From these estimates we conclude that

$$\sum_0^{\infty} [S(t2^{-k-1})f - S(t2^{-k})f]$$

converges in norm $\| \cdot \|_Z$ to an element of Z which has to be $f - S(t)f$, since $\lim_{k\to\infty} \| S(t2^{-k})f - f \|_X = 0$. Hence $f \in Z$, and furthermore, for $t = 1$,

$$\begin{aligned}
\| f \|_Z &\leqslant \| S(1)f \|_Z + [2D_Z(1 + m_z) + 4] \, \Phi_{z,1}[\| U(t)f - f \|_X] \\
&\leqslant \textit{const.} \, \| f \|_{z,1;\mathscr{U}}
\end{aligned}$$

in view of the fact that

$$\begin{aligned}
\| S(1)f \|_Z &\leqslant 2 \int_{1/2}^1 \| U(v)f \|_X \, dv + 2 \int_{1/2}^1 | U(v)f |_Z \, dv \\
&\leqslant M_{\mathscr{U}} \| f \|_X + D_Z \, z^{-1}(1/2) \, \| f \|_X \, .
\end{aligned}$$

This proves the theorem.

A converse result is given by

THEOREM 4. *Let $\mathscr{U}$ be an approximation process on X satisfying the Jackson and Bernstein-type inequalities* (2.4), (2.5) *of order $y(t)$ on X with respect to Y. Furthermore, let $z(t)$ be given as in Theorem* 3 *and such that $\Phi_{z,1}$ is a function seminorm upper-bounded by a constant and lower-bounded by $y(t)$. Then the inclusion relation* (3.4) *for a normal Banach subspace Z of X implies that $\mathscr{U}$ satisfies a Bernstein-type inequality* (3.3) *of order $z(t)$ on X with respect to Z.*

Proof. On account of (1.11) and (2.7), we have, by assumption,

$$Y < (X, Y; \Phi_{z,1}^{(y)}) = X_{z,1;\mathscr{U}} \subset Z,$$

so that $U(t)f \in \mathscr{M}(Z)$ for every $f \in Z$. Furthermore, (3.4) yields, for some constant D_Z',

$$\begin{aligned} \| U(t)f \|_Z &\leqslant D_Z' \left[\| f \|_X + \int_0^1 \| [U(s) - I]\, U(t)f \|_X z^{-1}(s)\, s^{-1}\, ds \right] \\ &\leqslant D_Z' \| f \|_X \left[1 + M_{\mathscr{U}}(1 + M_{\mathscr{U}}) \int_t^1 z^{-1}(s)\, s^{-1}\, ds \right. \\ &\qquad \left. + C_Y D_Y y^{-1}(t) \int_0^t y(s)\, z^{-1}(s)\, s^{-1}\, ds \right]. \end{aligned}$$

The latter inequality holds since, by (2.1),

$$\| [U(s) - I]\, U(t)f \|_X \leqslant M_{\mathscr{U}}(1 + M_{\mathscr{U}}) \qquad (t \leqslant s < 1),$$

and by (2.5), (2.6),

$$\begin{aligned} \| [U(s) - I]\, U(t)f \|_X &\leqslant C_Y y(s) \mid U(t)f \mid_Y \\ &\leqslant C_Y D_Y y(s)\, y^{-1}(t) \| f \|_X \qquad (0 < s < t). \end{aligned}$$

Now $\Phi_{z,1}$ is upper-bounded by a constant and lower-bounded by $y(t)$. Replacing $\Omega(t)$ and $z(t)$ in conditions (1.9), (1.10) by $z(t)$ and 1, respectively, it follows that there is a constant $D_Z > 0$ such that

$$\mid U(t)f \mid_Z \leqslant \| U(t)f \|_Z \leqslant D_Z \| f \|_X\, z^{-1}(t),$$

which was to be shown.

As an immediate consequence of Theorems 2, 3, and 4 and relation (2.7), we have

COROLLARY 1. *Let X, Y and $\mathscr{U}$ be given as in Theorem 4. Furthermore, let $z(t)$ be monotone increasing with $0 < z(t) \leqslant z(1) = 1$ and $z(t) \leqslant m_z z(t/2)$ for $t \in (0, 1]$ and such that $\Phi_{z,1}$ and $\Phi_{z,\infty}$ are regular function seminorms, both upper-bounded by a constant and lower-bounded by $y(t)$. Then $\mathscr{U}$ satisfies Jackson and Bernstein-type inequalities of order $z(t)$ on X with respect to a normal Banach subspace Z of X if and only if*

$$(X, Y; \Phi_{z,1}^{(y)}) \subset Z \subset (X, Y; \Phi_{z,\infty}^{(y)}). \tag{3.5}$$

This shows that if one has Jackson and Bernstein-type inequalities for $\mathscr{U}$ of order $y(t)$ on X with respect to Y, the problem of finding such inequalities of "lower" order $z(t)$ is reduced to that of finding certain intermediate spaces Z of X and Y satisfying (3.5). The applications in Section 4 will show that this is a practical procedure.

A few words about this corollary as an interpolation theorem. Rewrite condition (2.1) and the Jackson and Bernstein-type inequalities (2.5), (2.6) in the form

$$\| U(t) - I \|_{[X,X]} \leqslant M_{\mathscr{U}} + 1, \qquad \| U(t) \|_{[X,X]} \leqslant M_{\mathscr{U}} ;$$
$$\| U(t) - I \|_{[Y,X]} \leqslant C_Y y(t), \qquad \| U(t) \|_{[X,Y]} \leqslant D_Y y^{-1}(t),$$

using the notation $\| U \|_{[X_1,X_2]} = \sup_{f \in X_1} \| Uf \|_{X_2} / \| f \|_{X_1}$ for any linear operator U on a Banach space X_1 into a Banach space X_2. Now note that the first two inequalities could be interpreted as Jackson and Bernstein-type inequalities of order t^0 on X with respect to X. The "if" part of Corollary 1 states that, in case $y(t) = t^\alpha$, $\alpha > 0$,

$$\| U(t) - I \|_{[Z,X]} \leqslant C_Z\, t^\beta, \qquad \| U(t) \|_{[X,Z]} \leqslant D_Z\, t^{-\beta}$$

for intermediate values β, $0 < \beta < \alpha$, and corresponding intermediate spaces Z satisfying (3.5), since $\Phi_{\beta,1}$ and $\Phi_{\beta,\infty}$ are upper-bounded by 1 and lower-bounded by $y(t) = t^\alpha$. Thus, one could interpret the conclusion of the corollary as an interpolation of the above inequatilies for X and Y of order 1 and t^α to Jackson and Bernstein-type inequalities of intermediate order t^β, $0 < \beta < \alpha$, on X with respect to the intermediate spaces Z satisfying (3.5).

We remark that in this form the "if" part of Corollary 1 could also be established by a general interpolation theorem of Riesz–Thorin type for linear operators on K-intermediate spaces (compare Peetre [17, p. 18]).

The next interesting question is under what conditions does $\mathscr{U}$ satisfy

Jackson and Bernstein-type inequalities on Z with respect to Y, instead of on X with respect to Y.

THEOREM 5. *Let $\mathscr{U}$ be an approximation process on X defined by* (2.1)–(2.3).

(a) *If $\mathscr{U}$ satisfies a Jackson-type inequality of order $y(t)$ on X with respect to Y and a Bernstein-type inequality of order $z(t)$ on X with respect to Z, where $\Phi_{z,1}$ is lower-bounded by $y(t)$, i.e.,*

$$\int_0^t y(u)\, z^{-1}(u)\, u^{-1}\, du = O[y(t)\, z^{-1}(t)] \tag{3.6}$$

by (1.10), *then $Y \subset Z$ and*

$$\| U(t)f - f \|_Z \leqslant C(Z, Y)\, y(t)\, z^{-1}(t)\, \| f \|_Y \qquad (f \in Y), \tag{3.7}$$

i.e., $\mathscr{U}$ satisfies a Jackson-type inequality of (*reduced*) *order $y(t)\, z^{-1}(t)$ on Z with respect to Y.*

(b) *If $\mathscr{U}$ satisfies a Bernstein-type inequality of order $y(t)$ on X with respect to Y and a "Jackson-type" one of order $z(t)$ on X with respect to Z, where $\Phi_{z,\infty}$ is lower-bounded by $y(t)$, i.e.,*

$$\int_t^1 y^{-1}(u)\, z(u)\, u^{-1}\, du = O[y^{-1}(t)\, z(t)] \tag{3.8}$$

by (1.8), *then*

$$\| U(t)f \|_Y \leqslant D(Z, Y)(y^{-1}(t)\, z(t)\, \| f \|_Z + \| f \|_X) \qquad (f \in Z), \tag{3.9}$$

i.e., $\mathscr{U}$ satisfies a "Bernstein-type" inequality of (*reduced*) *order $y(t)\, z^{-1}(t)$ on Z with respect to Y.*

Proof. We carry on the estimates begun in Theorem 3 for $f \in Y$, by the Jackson-type inequality (2.5) which gives, by (3.6),

$$\begin{aligned}
&\sum_{k=0}^{\infty} \| S(t2^{-k}f) - S(t2^{-k-1})f \|_Z \\
&\quad \leqslant [2D_Z(1 + m_z) + 4] \int_0^t z^{-1}(v)\, \| U(v)f - f \|_X\, v^{-1}\, dv \\
&\quad \leqslant \| f \|_Y C_Y [2D_Z(1 + m_z) + 4] \int_0^t y(v)\, z^{-1}(v)\, v^{-1}\, dv \\
&\quad \leqslant C_1(Z, Y)\, y(t)\, z^{-1}(t)\, \| f \|_Y .
\end{aligned}$$

In a similar manner, we obtain

$$\| U(t)f - S(t)f \|_Z \leqslant 2D_Z C_Y \| f \|_Y \int_{t/2}^{t} [y(v)\, z^{-1}(v/2) + y(v/2)\, z^{-1}(v)]\, v^{-1}\, dv$$
$$\leqslant C_2(Z, Y)\, y(t)\, z^{-1}(t)\, \| f \|_Y$$

for $f \in Y$. Then, arguing as in Theorem 3, it follows that, for $f \in Y$,

$$U(t)f - S(t)f + \sum_{k=0}^{\infty} [S(t2^{-k})f - S(t2^{-k-1})f] = U(t)f - f$$

is an element of Z. This implies $Y \subset Z$, and with

$$C(Z, Y) = C_1(Z, Y) + C_2(Z, Y)$$

it follows that

$$\| U(t)f - f \|_Z \leqslant C(Z, Y)\, y(t)\, z^{-1}(t)\, \| f \|_Y \qquad (f \in Y).$$

To prove (3.9), we use the estimate of part (b) of Theorem 1 and then the Jackson-type inequality (3.1) for $f \in Z$ to deduce

$$\| U(t)f \|_Y \leqslant D_Y(3 + m_y) \int_{t/2}^{1} y^{-1}(v)\, \| U(v)f - f \|_X\, v^{-1}\, dv$$
$$+ D_Y m_y\, y^{-1}(t)\, \| U(t)f - f \|_X + D_Y \| f \|_X\, y^{-1}(1/2)$$
$$\leqslant D_Y C_Z(3 + m_y)\, \| f \|_Z \left\{ \int_{t/2}^{1} y^{-1}(v)\, z(v)\, v^{-1}\, dv + y^{-1}(t)\, z(t) \right\}$$
$$+ D_Y \| f \|_Y\, y^{-1}(1/2).$$

Applying (3.8), this yields the assertion (3.9).

We now come to our main theorem which extends Theorem 1 and includes the already mentioned assertion of reduction type.

THEOREM 6. *Let Y, Z be normal Banach subspaces of the Banach space X, let $\mathscr{U}$ and $U(t)f$ be as in Theorem* 1, *and let Φ be a regular function seminorm.*

(a) *If $\mathscr{U}$ satisfies a Jackson-type inequality of order $y(t)$ on X with respect to Y and Φ is upper-bounded by a constant, then*

$$\Phi[y(t)\, \| U(t)f \|_Y] < \infty \Rightarrow \Phi[\| U(t)f - f \|_X] < \infty.$$

If, in addition, $\mathscr{U}$ satisfies a Bernstein-type inequality of order $z(t)$ on X with *respect to Z such that $\Phi_{z,1}$ is lower-bounded by $y(t)$, and Φ is upper-bounded by $z(t)$, then*

$$\Phi[y(t) \mid U(t)f\mid_Y] < \infty \Rightarrow f \in Z, \Phi[z(t) \mid U(t)f - f\mid_Z] < \infty. \tag{3.10}$$

(b) *If $\mathscr{U}$ satisfies a Bernstein-type inequality of order $y(t)$ on X with respect to Y and Φ is lower-bounded by $y(t)$ and upper-bounded by a constant, then*

$$\Phi[\| U(t)f - f\|_X] < \infty \Rightarrow \Phi[y(t) \mid U(t)f\mid_Y] < \infty.$$

If, in addition, $\mathscr{U}$ satisfies a Jackson-type inequality of order $z(t)$ on X with respect to Z such that $\Phi_{z,\infty}$ is lower-bounded by $y(t)$, then

$$f \in Z, \Phi[z(t) \mid U(t)f - f\mid_Z] < \infty \Rightarrow \Phi[y(t) \mid U(t)f\mid_Y] < \infty. \tag{3.11}$$

(c) *If $\mathscr{U}$ satisfies Jackson and Bernstein-type inequalities of orders $y(t)$, $z(t)$ on X with respect to Y, Z (i.e., $\mathscr{U}$ satisfies inequalities* (2.5), (2.6), (3.1) *and* (3.3)), *where $\Phi_{z,1}$ and $\Phi_{z,\infty}$ are lower-bounded by $y(t)$, then for every regular function seminorm which is lower-bounded by $y(t)$ and upper-bounded by $z(t)$, the above three assertions are equivalent to each other as well as to*

$$\Phi[K(y(t), f; X, Y)] < \infty \qquad \text{for} \quad f \in X.$$

Proof. The first half of (a) and of (b) have already been proved in Theorem 1. To prove the second half of (a), we proceed as in the corresponding part of Theorem 1, using an analogue of (2.8) where X is replaced by Z and (2.5) by the inequality (3.7), in accordance with Theorem 5. We have

$$\begin{aligned}
&|U(t)f - S(t)f\mid_Z + \sum_{k=0}^{\infty} \mid S(t2^{-k})f - S(t2^{-k-1})f\mid_Z \\
&\leqslant 2\int_{t/2}^{t} \mid U(t)f - U(v)f\mid_Z v^{-1}\,dv + 2\int_0^t \mid U(v)f - U(v/2)f\mid_Z v^{-1}\,dv \\
&\leqslant 2C(Z, Y)\int_{t/2}^{t} [y(t)\,z^{-1}(t) \mid U(v)f\mid_Y + y(v)\,z^{-1}(v) \mid U(t)f\mid_Y]\, v^{-1}\,dv \\
&\qquad + 2C(Z, Y)\int_0^t [y(v)\,z^{-1}(v) \mid U(v/2)f\mid_Y \\
&\qquad\qquad + y(v/2)z^{-1}(v/2) \mid U(v)f\mid_Y]\, v^{-1}\,dv \\
&\leqslant 2C(Z, Y)\Big[m_z\, y(t)\,z^{-1}(t) \mid U(t)f\mid_Y \\
&\qquad + (2m_y + m_z)\int_0^t y(v)\,z^{-1}(v) \mid U(v)f\mid_Y v^{-1}\,dv\Big].
\end{aligned}$$

Furthermore, we obtain directly from part (a) of Theorem 1, using $1 \leqslant z^{-1}(t)$ for $t \in (0, 1]$,

$$\| U(t)f - S(t)f \|_X + \sum_{k=0}^{\infty} \| S(t2^{-k})f - S(t2^{-k-1})f \|_X$$
$$\leqslant 2C_Y \left[(2m_y + 1) \int_0^t y(v)\, z^{-1}(v) \mid U(v)f \mid_Y v^{-1} dv + y(t)\, z^{-1}(t) \mid U(t)f \mid_Y\right].$$

Using both estimates, it follows from $\Phi[y(t) \mid U(t)f \mid_Y] < \infty$ by the upper-boundedness of Φ by $z(t)$ that

$$\Phi \left[z(t) \| U(t)f - S(t)f \|_Z + z(t) \sum_{k=0}^{\infty} \| S(t2^{-k})f - S(t2^{-k-1})f \|_Z\right] < \infty.$$

By the regularity (1.3) of Φ this implies that

$$\sum_0^{\infty} \| S(t2^{-k})f - S(t2^{-k-1})f \|_Z < \infty,$$

and hence

$$\sum_0^{\infty} [S(t2^{-k})f - S(t2^{-k-1})f] = (S(t)f - f) \in Z$$

for almost all $t \in (0, 1]$. But then $f \in Z$, and assertion (3.10) follows since, for $f \in Z$,

$$\mid U(t)f - f \mid_Z \leqslant \| U(t)f - S(t)f \|_Z + \sum_{k=0}^{\infty} \| S(t2^{-k})f - S(t2^{-k-1})f \|_Z .$$

To prove (3.11) we proceed as in part (b) of Theorem 1, using an analogue of (2.9), where X is replaced by Z and (2.6) by the inequality (3.9) according to Theorem 5:

$$\begin{aligned}
\mid U(t)f \mid_Y \leqslant 2D(Z, Y) \int_{t/2}^{1} & [y^{-1}(t)\, z(t) \mid U(v)f - f \mid_Z \\
& + y^{-1}(v)\, z(v) \mid U(t)f - f \mid_Z]\, v^{-1}\, dv \\
& + 2D(Z, Y) \int_{t/2}^{t} [\| U(v)f - f \|_X + \| U(t)f - f \|_X]\, v^{-1}\, dv \\
& + D_Y \int_{1/2}^{1} \| f \|_X\, y^{-1}(v)\, v^{-1}\, dv + D(Z, Y) \int_t^1 [y^{-1}(v)\, z(v) \mid U(v/2)f - f \mid_Z \\
& + y^{-1}(v/2)\, z(v/2) \mid U(v)f - f \mid_Z]\, v^{-1}\, dv \\
& + D(Z, Y) \int_t^1 [\| U(v/2)f - f \|_X + \| U(v)f - f \|_X]\, v^{-1}\, dv.
\end{aligned}$$

Using the various properties of $y(t)$, $z(t)$ and (2.1), we can estimate further, as in Theorem 1, part (b):

$$\begin{aligned}|U(t)f|_Y \leqslant 2D(Z, Y)\Big[m_z \int_{t/2}^{1} & y^{-1}(v)\, z(v)\, |U(v)f - f|_Z\, v^{-1}\, dv \\ & + m_y y^{-1}(t)\, z(t)\, |U(t)f - f|_Z\Big] \\ & + \|f\|_X[D_Y y^{-1}(1/2) + 4(M_{\mathscr{U}} + 1)\, D(Z, Y)] \\ & + D(Z, Y)(m_z + m_y) \int_{t/2}^{1} y^{-1}(v)\, z(v)\, |U(v)f - f|_Z\, v^{-1}\, dv \\ & + 2D(Z, Y) \int_{t/2}^{1} \|U(v)f - f\|_X\, v^{-1}\, dv.\end{aligned}$$

Since

$$\begin{aligned}\Phi\Big[y(t) \int_{t/2}^{1} \|U(v)f - f\|_X\, v^{-1}\, dv\Big] &\leqslant m_y B(\Phi, y)\, \Phi[y(t/2)\, \|U(t/2)f - f\|_X] \\ &\leqslant m_y B(\Phi, y)(M_{\mathscr{U}} + 1)\, \Phi[y(t)],\end{aligned}$$

we can now conclude, just as in Theorem 1, that $\Phi[y(t)\, |U(t)f|_Y] < \infty$ provided $\Phi[z(t)\, |U(t)f - f|_Z] < \infty$ and $\Phi[y(t)] < \infty$, the latter relation being satisfied by assumption and Lemma 1(c).

Finally, part (c) is an immediate consequence of the preceding if one observes that Φ is upper-bounded by a constant if it by $z(t)$ (see Lemma 1(a)).

Let us concretize this theorem in the representative case $\Phi = \Phi_{\Omega,q}$.

COROLLARY 2. *Let $\mathscr{U}$ be a linear approximation process satisfying Jackson and Bernstein-type inequalities of orders $y(t)$, $z(t)$ on X with respect to Y, Z, and let $z(t)$, $y(t)$ satisfy* (3.6) *and* (3.8). *If the function seminorm $\Phi_{\Omega,\infty}$ satisfies* (1.7) *and* (1.8), *then the following assertions are equivalent for $t \to 0+$:*

(a) $$\|U(t)f - f\|_X = O[\Omega(t)],$$

(b) $$|U(t)f|_Y = O[y^{-1}(t)\, \Omega(t)],$$

(c) $$f \in Z,\ |U(t)f - f|_Z = O[z^{-1}(t)\, \Omega(t)],$$

(d) $$K(y(t), f; X, Y) = O[\Omega(t)].$$

If, in addition, conditions (1.9) *and* (1.10) *hold, then*

(a)′ $$\int_0^1 [\Omega^{-1}(t)\, \|U(t)f - f\|_X]^q\, t^{-1}\, dt < \infty,$$

(b)′ $$\int_0^1 [\Omega^{-1}(t)\, y(t)\, |U(t)f|_Y]^q\, t^{-1}\, dt < \infty,$$

$$\text{(c)}' \quad f \in Z, \qquad \int_0^1 [\Omega^{-1}(t)\, z(t) \mid U(t)f - f\,|_Z]^q\, t^{-1}\, dt < \infty,$$

$$\text{(d)}' \qquad \int_0^1 [\Omega^{-1}(t)\, K(y(t), f;\, X,\, Y)]^q\, t^{-1}\, dt < \infty$$

are equivalent for $1 \leqslant q < \infty$.

Let us remark that for the particular choice $\Omega(t) = t^\theta$, $y(t) = t^l$ and $z(t) = t^k$, the assumptions of this corollary are equivalent to $0 \leqslant k < \theta < l$.

Let us also consider a discrete version of this corollary. By this we mean that $\mathscr{U}$ is defined by a sequence $\mathscr{V} = \{V_n\}_1^\infty$ of operators V_n of $\mathscr{E}(X)$ which satisfy the basic conditions (2.1)–(2.3). Setting $U(t)f = V_{[1/t]}f$, this amounts to the conditions

$$\| V_n f\|_X \leqslant M'_{\mathscr{V}} \|f\|_X \qquad (n \in \mathbb{N}), \tag{2.1$'$}$$

$$V_n V_m = V_m V_n \qquad (m, n \in \mathbb{N}), \tag{2.2$'$}$$

$$\lim_{n\to\infty} \| V_n f - f\|_X = 0 \tag{2.3$'$}$$

for every $f \in X$, where $\mathbb{N}$ denotes the set of all positive integers. The Jackson and Bernstein-type inequalities of orders $y(t)$, $z(t)$ on X with respect to Y and Z take then the particular form

$$\| V_n f - f\|_X \leqslant C'_Y y(1/n)\, | f |_Y \qquad (f \in Y), \tag{2.5$'$}$$

$$V_n f \in Y, \quad | V_n f |_Y \leqslant D'_Y y^{-1}(1/n)\, \|f\|_X \qquad (f \in X), \tag{2.6$'$}$$

$$\| V_n f - f\|_X \leqslant C'_Z z(1/n)\, | f |_X \qquad (f \in Z), \tag{3.1$'$}$$

$$V_n f \in Z, \quad | V_n f |_Z \leqslant D'_Z z^{-1}(1/n)\, \|f\|_X \qquad (f \in X) \tag{3.3$'$}$$

for all $n \in \mathbb{N}$. Note that here the assumptions $V(t)f \in \mathscr{M}(Y)$, $V(t)f \in \mathscr{M}(Z)$ reduce to $V_n f \in Y$ and $V_n f \in Z$, respectively, for all $n \in \mathbb{N}$.

Now we can state the discrete version of the above corollary (the case $q = \infty$, for simplicity).

COROLLARY 3. *Let* $\{V_n\}_1^\infty$ *be a sequence of operators of* $\mathscr{E}(X)$ *satisfying* (2.1)$'$–(2.6)$'$ *and* (3.1)$'$, (3.3)$'$. *Let* $\Omega(t)$, $y(t)$, $z(t)$ *satisfy, furthermore, the conditions* (3.6), (3.8), (1.7), *and* (1.8). *Then the following assertions are equivalent, for* $n \to \infty$:

$$\text{(a)} \qquad \| V_n f - f\|_X = O[\Omega(1/n)],$$

$$\text{(b)} \qquad | V_n f |_Y = O[y^{-1}(1/n)\, \Omega(1/n)],$$

$$\text{(c)} \qquad f \in Z, \quad | V_n f - f |_Z = O[z^{-1}(1/n)\, \Omega(1/n)],$$

$$\text{(d)} \qquad K(y(1/n), f;\, X,\, Y) = O[\Omega(1/n)].$$

This result extends our previous ones in [9] and contains those announced in [10] for the particular instance $y(t) = t^l$, $z(t) = t^k$ and $\Omega(t) = t^\theta$.

4. Applications

4.1. *Holomorphic Semigroup Operators*

Let us apply the results of the preceding section to approximation processes generated by a one-parameter family $\mathscr{T} = \{T(t) : 0 \leqslant t < \infty\}$ of equibounded semigroup operators in $\mathscr{E}(X)$ of class (C_0), with $M_{\mathscr{T}} = \sup_t \| T(t)\| < \infty$. Then conditions (2.1)–(2.3) are satisfied for the family $\mathscr{T}_r = \{T_r(t) = I - [I - T(t)]^r : 0 \leqslant t \leqslant 1\}$, $r \in \mathbb{N}$, for which $T_1(t) = T(t)$. The problem now is to find subspaces of X for which T_r satisfies Jackson and Bernstein-type inequalities. As the following will show this can easily be achieved by considering the domain $D((-A)^\gamma)$ of the fractional power $(-A)^\gamma$, $0 < \gamma < r$, of the infinitesimal generator-A.

Definition 9. An element $f \in X$ belongs to $D((-A)^\gamma)$, $0 < \gamma < r$, if and only if

$$\operatorname*{s\text{-}lim}_{\epsilon \to 0} c_{\gamma,r}^{-1} \int_\epsilon^\infty t^{-\gamma}[I - T(t)]^r f\, t^{-1}\, dt \equiv (-A)^\gamma f$$

exists, where

$$c_{\gamma,r} = \int_0^\infty t^{-\gamma}(1 - e^{-t})^r\, t^{-1}\, dt.$$

Concerning this definition and other equivalent ones, connected with the names of Phillips and Balakrishnan, see Westphal [21]. In case γ is a positive integer, it coincides with the usual one by a result of Lions–Peetre [15]. Furthermore, it follows for $f \in D((-A)^\gamma)$ (see [21]) that

$$[I - T(t)]^r f = t^\gamma \Gamma(\gamma)^{-1} \int_0^\infty p_{\gamma,r}(u)\, T(tu)\, (-A)^\gamma f\, du \quad (0 < \gamma \leqslant r,\ t > 0), \tag{4.1}$$

where $p_{\gamma,r}(u)$ is a function of $L_1(0, \infty)$ defined by its Laplace transform

$$\Gamma(\gamma)(1 - e^{-\lambda})^r = \lambda^\gamma \int_0^\infty e^{-\lambda u} p_{\gamma,r}(u)\, du.$$

Lemma 6. *Let $\{T(t) : 0 \leqslant t < \infty\}$ be the above-mentioned semigroup of* *operators. Then for each γ, β with $0 < \gamma < \beta$ the inclusions (recall Section 1 for notation)*

$$(X, D((-A)^\beta); \Phi_{\gamma,1}^{(\beta)}) \subset D((-A)^\gamma) \subset (X, D((-A)^\beta); \Phi_{\gamma,\infty}^{(\beta)}) \tag{4.2}$$

are valid with respect to the norm $\|f\|_X + \|(-A)^\gamma f\|_X$ defined for $f \in D((-A)^\gamma)$. Furthermore, if the semigroup is also holomorphic, the approximation process $\mathscr{T}_r$ satisfies Jackson and Bernstein-type inequalities of order t^γ, $0 < \gamma \leqslant r$, on X with respect to $D((-A)^\gamma)$, i.e.,

$$\| T_r(t)f - f\|_X \leqslant \left[\Gamma^{-1}(\gamma)\, M_{\mathscr{T}} \int_0^\infty |p_{\gamma,r}(u)|\, du\right] t^\gamma \|(-A)^\gamma f\|_X \qquad (f \in D((-A)^\gamma)),$$

$$\|(-A)^\gamma\, T_r(t) f\|_X \leqslant D_{r,\gamma} t^{-\gamma} \|f\|_X \qquad (f \in X).$$

Proof. From Definition 9 we conclude by (1.14):

$$\begin{aligned}\|(-A)^\gamma f\|_X &\leqslant c_{\gamma,r}^{-1} \int_0^\infty t^{-\gamma} \| [I - T(t)]^r f\|_X\, t^{-1}\, dt \\ &\leqslant c_{\gamma,r}^{-1}(M_{\mathscr{T}} + 1)^r \|f\|_X + c_{\gamma,r}^{-1} \Phi_{\gamma,1}[\omega_r(t, f; \mathscr{T})] < \infty\end{aligned}$$

for $f \in (X, D(A^r); \Phi_{\gamma,1}^{(r)})$, $r > \gamma$; and from relation (4.1):

$$\Phi_{\gamma,\infty}[\omega_r(t, f; \mathscr{T})] \leqslant \left[\Gamma^{-1}(\gamma)\, M_{\mathscr{T}} \int_0^\infty |p_{\gamma,r}(u)|\, du\right] \|(-A)^\gamma f\|_X \tag{4.3}$$

for $f \in D((-A)^\gamma)$, so that, again by (1.14), relation (4.2) is established for integral $\beta = r$. But (4.2) also follows for nonintegral β since (see Berens [2, p. 46]),

$$(X, D((-A)^\beta); \Phi_{\gamma,q}^{(\beta)}) = (X, D(A^r); \Phi_{\gamma,q}^{(r)}) \qquad (1 \leqslant q \leqslant \infty; 0 < \gamma < \beta < r). \tag{4.4}$$

In order to show the existence of Bernstein-type inequalities we consider first the case $\gamma = r$ of highest order.[5] Then by the holomorphic property of $T(t)$, the range $R[T(t)]$ is contained in $D(A^r)$, and $T(t)f$ as well as $T_r(t)f$ are strongly continuous (and hence measurable) for each $f \in X$ with respect to the Banach norm $\|f\|_X + \|A^r f\|_X$ for $D(A^r)$. Furthermore, the Cauchy integral formula (see [4, p. 17, 292]) yields, for some $\alpha > 0$, $N_r > 0$,

$$\| A^r T_r(t)\| = \left\| \frac{1}{2\pi i} \int_{|\zeta - t| = t \sin\alpha} \frac{T_r(\zeta)\, d\zeta}{(\zeta - t)^{r+1}} \right\| \leqslant N_r t^{-r},$$

[5] This is in fact the highest possible order since $\| T_r(t)f - f\|_X = o(t^r)$, $t \to 0+$, implies $A^r f = 0$ or $T_r(t)f - f = 0$ for all $t > 0$. Concerning this "saturation" theorem see [4, p. 102].

so that a Bernstein-type inequality of order t^r on X with respect to $D(A^r)$ is satisfied. The Bernstein-type inequality of order t^γ, $0 < \gamma < r$, is now an immediate consequence of Corollary 1 since its assumptions are fulfilled in view of (4.2) and $0 < \gamma < r$. The corresponding Jackson-type inequality follows directly from (4.3).

Choosing $Y = D(A^r)$, $y(t) = t^r$ and $Z = D((-A)^\gamma)$, $z(t) = t^\gamma$, $0 < \gamma < r$, the hypotheses of Corollary 2 are satisfied for the family $\mathscr{T}_r$. Its application therefore yields

THEOREM 7. *Let $\mathscr{T}_r$ be the approximation process*

$$\{T_r(t) = I - [I - T(t)]^r : 0 \leqslant t \leqslant 1\}, \qquad r \in \mathbb{N},$$

where $\{T(t) : 0 \leqslant t < \infty\}$ is an equi-bounded family of holomorphic semigroup operators of class (C_0). Furthermore, let $\Omega(t)$ be a nondecreasing function on $(0, 1]$ satisfying for $0 < \gamma < r$, $t \to 0+$,

$$\int_0^t \Omega(u)\, u^{-\gamma-1}\, du = O[t^{-\gamma}\Omega(t)], \qquad \int_t^1 \Omega^{-1}(u)\, u^{\gamma-1}\, du = O[t^\gamma \Omega^{-1}(t)], \tag{4.5}$$

$$\int_t^1 \Omega(u)\, u^{-r-1}\, du = O[t^{-r}\Omega(t)], \qquad \int_0^t \Omega^{-1}(u)\, u^{r-1}\, du = O[t^r \Omega^{-1}(t)]. \tag{4.6}$$

If $f \in X$, the following assertions are equivalent for each q, $1 \leqslant q \leqslant \infty$:

(a) $$\int_0^1 [\Omega^{-1}(t) \| T_r(t)f - f \|_X]^q\, t^{-1}\, dt < \infty,$$

(b) $$f \in D((-A)^\gamma), \quad \int_0^1 [\Omega^{-1}(t)\, t^\gamma \|(-A)^\gamma T_r(t)f - (-A)^\gamma f\|_X]^q\, t^{-1}\, dt < \infty,$$

(c) $$\int_0^1 [\Omega^{-1}(t)\, t^r \| A^r T_r(t) f\|_X]^q\, t^{-1}\, dt < \infty,$$

(d) $$\int_0^1 [\Omega^{-}1(t)\, K(t^r, f; X, D(A^r))]^q\, t^{-1}\, dt < \infty.$$

Let us remark that the equivalence (a) $\Leftrightarrow$ (d) has already been shown in [17, 18] for functions $\Omega(t)$ which are essentially submultiplicative. However, our theory is not limited to this case (compare footnote 3). In case $\Omega(t) = t^\theta$, $0 < \theta < r$, γ an integer, (a) is equivalent to (b) also by the reduction theorem of relation (1.15) since $(-A)^\gamma$ commutes with $T_r(t)$, i.e.,

$$(-A)^\gamma\, T_r(t)f - (-A)^\gamma f = -[I - T(t)]^r(-A)^\gamma f \qquad (f \in D((-A)^\gamma)). \tag{4.7}$$

Assertion (c) should be compared with the following equivalent statement:

$$\text{(c)}' \qquad \int_0^\infty [t^{r-\theta} \| A^r T(t) f \|_X]^q \, t^{-1} \, dt < \infty,$$

which is of Zamansky's type (see [4, p. 210] and the references given there).

4.2. *Resolvent Operators*

As a second example of a linear approximation process, we consider the family of resolvent operators $\{\lambda R(\lambda; A), \lambda > 0\}$ of the infinitesimal generator A of an equi-bounded family of semigroup operators $\{T(t) : 0 \leqslant t < \infty\}$ of class (C_0), namely,

$$\lambda R(\lambda; A) f = \lambda \int_0^\infty e^{-\lambda t} T(t) f \, dt \qquad (\lambda > 0; f \in X).$$

Setting $R(t) = \lambda R(\lambda; A)$ with $t = \lambda^{-1}$, this family of operators satisfies the conditions (2.1)–(2.3) for $t \in (0, 1]$ or $\lambda \in [1, \infty)$ since $\lambda R(\lambda; A) f$ is strongly continuous and uniformly bounded for $\lambda \geqslant 1$, commutative in view of the resolvent equation

$$R(\lambda_1 ; A) - R(\lambda_2 ; A) = (\lambda_2 - \lambda_1) R(\lambda_1 ; A) R(\lambda_2 ; A),$$

and convergent to f for every $f \in X$ since ([4, p. 131]),

$$\lim_{\lambda \to \infty} \| \lambda R(\lambda; A) f - f \|_X = 0.$$

In order to establish Jackson and Bernstein-type inequalities, we make use of the defining relations for the resolvent $R(\lambda; A)$, namely,

$$\begin{aligned} &\text{(i)} \qquad && (\lambda I - A) R(\lambda; A) f = f \qquad (f \in X), \\ &\text{(ii)} \qquad && R(\lambda; A)(\lambda I - A) f = f \qquad (f \in D(A)). \end{aligned} \tag{4.8}$$

By (ii) it follows that, for $\lambda \geqslant 1$,

$$\| \lambda R(\lambda; A) f - f \|_X = \| R(\lambda; A) Af \|_X \leqslant \lambda^{-1} M_{\mathscr{R}} \| Af \|_X \qquad (f \in D(A)),$$

and by (i), $\lambda R(\lambda; A) f \in \mathscr{M}(D(A))$, and

$$\| A(\lambda R(\lambda; A) f) \|_X = \lambda \| \lambda R(\lambda; A) f - f \|_X \leqslant \lambda (M_{\mathscr{R}} + 1) \| f \|_X \qquad (f \in X),$$

where $M_{\mathscr{R}} = \sup_{\lambda \geqslant 1} \| \lambda R(\lambda; A) \| < \infty$.

Hence the process $\mathscr{R} = \{R(t) : R(t) = \lambda R(\lambda; A), \lambda = t^{-1}, 0 < t \leqslant 1\}$ satisfies Jackson and Bernstein-type inequalities of order t on X with respect to $D(A)$. Since $\| \lambda R(\lambda; A)f - f\| = o(\lambda^{-1})$, $\lambda \to \infty$, for $f \in X$ implies $Af = 0$ or $\lambda R(\lambda; A)f - f = 0$ (see [4, p. 153]), this order is the best possible or "saturation" order of the approximation process $\mathscr{R}$ on X. An application of the "sufficiency" part of Corollary 1 to the family $\mathscr{R}$ gives Jackson and Bernstein-type inequalities of intermediate order t^γ, $0 < \gamma < 1$, on X with respect to $D((-A)^\gamma)$, since its assumptions are satisfied in view of (4.2) and $0 < \gamma < 1$.

We can now state

THEOREM 8. *Let $\mathscr{R}$ be the above approximation process. Furthermore, let $\omega(\lambda)$ be a positive nonincreasing function on $[1, \infty)$ satisfying for $0 < \gamma < 1$, $\lambda \to \infty$,*

$$\int_\lambda^\infty \omega(u)\, u^{\gamma-1}\, du = O[\lambda^\gamma \omega(\lambda)], \qquad \int_1^\lambda \omega^{-1}(u)\, u^{-1-\gamma}\, du = O[\lambda^{-\gamma}\omega^{-1}(\lambda)], \tag{4.9}$$

$$\int_1^\lambda \omega(u)\, du = O[\lambda\omega(\lambda)], \qquad \int_\lambda^\infty \omega^{-1}(u)\, u^{-2}\, du = O[\lambda^{-1}\omega^{-1}(\lambda)]. \tag{4.10}$$

If $f \in X$, the following assertions are equivalent for $1 \leqslant q \leqslant \infty$:

(a) $$\int_1^\infty [\omega^{-1}(\lambda) \| \lambda R(\lambda; A)f - f\|_X]^q\, \lambda^{-1}\, d\lambda < \infty,$$

(b) $f \in D((-A)^\gamma)$,

$$\int_1^\infty [\omega^{-1}(\lambda)\, \lambda^{-\gamma} \|(-A)^\gamma \lambda R(\lambda; A)f - (-A)^\gamma f\|_X]^q\, \lambda^{-1}\, d\lambda < \infty,$$

(c) $$\int_1^\infty [\omega^{-1}(\lambda) \| AR(\lambda; A)f\|_X]^q\, \lambda^{-1}\, d\lambda < \infty,$$

(d) $$\int_0^1 [\omega^{-1}(t^{-1})\, K(t, f; X, D(A)]^q\, t^{-1}\, dt < \infty,$$

(e) $$\int_0^1 [\omega^{-1}(t^{-1})\omega_1(t, f, \mathscr{T})]^q\, t^{-1}\, dt < \infty.$$

Proof. The equivalence of the first four assertions follows just as Theorem 7 by applying Corollary 2 to $\mathscr{R} = \{R(t) = \lambda R(\lambda; A) : 0 < t \leqslant 1, \lambda \in [1, \infty)\}$, $Y = D(A)$, $y(t) = t$ and $Z = D((-A)^\gamma)$, $z(t) = t^\gamma$, $0 < \gamma < 1$. We have to observe that in view of the transformation $\lambda = t^{-1}$ the assumptions (4.9), (4.10) coincide with (4.5), (4.6) for the nondecreasing function $\Omega(t) = \omega(1/t)$, and that this corollary yields, e.g., instead of (a), an assertion of type

$$\text{(a)}' \qquad \int_0^1 [\omega^{-1}(t^{-1}) \| R(t)f - f \|_X]^q \, t^{-1} \, dt < \infty,$$

which, however, is equivalent to (a).

The equivalence (d) $\Leftrightarrow$ (e) is established by relation (1.14).

Let us remark that in case $\omega(\lambda) = \lambda^{-\theta}$ the equivalence (a) $\Leftrightarrow$ (e) has already been shown in Butzer–Pawelke [6] for $q = \infty$, and by Berens [2] for $1 \leqslant q < \infty$. The equivalence with (b) could also have been established by the above-mentioned reduction theorem of relation (1.15) since $\lambda R(\lambda; A)$ commutes with $(-A)^\gamma$ just as $T_\gamma(t)$ does (compare (4.7)). Note further that for this particular approximation process $\mathscr{R}$ the equivalence (a) $\Leftrightarrow$ (c) is given in a trivial manner by the relation (4.8,i).

As examples of approximation processes generated by sequences of operators considered in Corollary 3, let us mention summation processes of Fourier series of 2π-periodic functions f belonging to $C_{2\pi}$ or to one of the Lebesgue spaces $L^p_{2\pi}$, $1 \leqslant p \leqslant \infty$. These are of the form

$$V_n(f; x) = \sum_{k=-n}^{n} \lambda_{k,n} f^\wedge(k) \, e^{ikx},$$

where $f^\wedge(k) = (1/(2\pi)) \int_{-\pi}^{\pi} f(u) \, e^{-iku} \, du$ is the kth Fourier coefficient of f ($k = 0, +1, +2, \ldots$), the summation factors $\lambda_{k,n}$ satisfying certain conditions. Various examples of such summation processes have been discussed from this standpoint in Butzer–Scherer [7, 10].

4.3. *Riesz-Means of the Fourier Inversion-Integral*

We conclude by considering an approximation process which belongs to none of the above categories of examples, namely, the Riesz-Means of the Fourier inversion-integral of functions $f \in L^p(-\infty, \infty)$, $1 \leqslant p \leqslant 2$. It is defined by

$$[R_{\gamma,\delta;\rho}(f)](x) = (\rho/\sqrt{2\pi}) \int_{-\infty}^{\infty} f(x-u) \, \chi_{\gamma,\delta}(\rho u) \, du \quad (\rho > 0, \, \rho \to \infty), \tag{4.11}$$

the kernel $\chi_{\gamma,\delta}$ being defined though

$$\chi^\wedge_{\gamma,\delta}(v) = \begin{cases} (1 - |v|^\gamma)^\delta, & |v| \leqslant 1 \\ 0, & |v| \geqslant 1 \end{cases} \qquad (\gamma, \delta > 0 \text{ fixed}),$$

$f^\wedge(v) = (1/\sqrt{2\pi}) \int_{-\infty}^{\infty} e^{-ivx} f(x) \, dx$ denoting the Fourier transform of an element $f \in L^p(-\infty, \infty)$. Setting $\rho = t^{-1}$, the family

$$\mathscr{R}_{\gamma,\delta} = \{R_{\gamma,\delta}(t)f = R_{\gamma,\delta;\rho}(f) : t \in (0, 1], \rho \in [1, \infty)\}$$

is a linear approximation process on $L^p(-\infty, \infty)$, satisfying (2.1)–(2.3) as well as Jackson and Bernstein-type inequalities of orders $\rho^{-\gamma}$ or t^γ on $L^p(-\infty, \infty)$ with respect to the space

$$[L^p]^{\{\gamma\}} = \{f \in L^p(-\infty, \infty) : \exists g \in L^p(-\infty, \infty) \text{ with } |v|^\gamma f^\wedge(v) = g^\wedge(v)\}.$$

This order is the highest possible (saturation order) of the family $\mathscr{R}_{\gamma,\delta}$.

Furthermore, it can be shown that $[L^p]^{\{\gamma\}} = D((-\tilde{A})^\gamma)$, where $\tilde{A}$ denotes the infinitesimal generator of the holomorphic contraction semigroup given by the singular integral of Cauchy–Poisson; thus

$$D(\tilde{A}) = \{f \in L^p(-\infty, \infty) : \exists g = \tilde{A}f \in L^p(-\infty, \infty) \text{ with } |v| f^\wedge(v) = g^\wedge(v)\}.$$

For all these facts we refer to Berens [2, pp. 82–83], Butzer–Nessel [5].

Now let A' be the infinitesimal generator of the ordinary translation group on $L^p(-\infty, \infty)$, i.e., $A'f = f'$ and

$$D(A') = \{f \in L^p(-\infty, \infty) : f' = g \in L^p(-\infty, \infty), ivf^\wedge(v) = g^\wedge(v)\}.$$

Denoting $D((-A')^\gamma)$ by $[L^p]^{(\gamma)}$, $(-A')^\gamma f$ by $f^{(\gamma)}$, and $(-\tilde{A})^\gamma f$ by $f^{\{\gamma\}}$ (the symbol for the fractional Riesz-derivative; for details see [5]), we have the following result concerning "intermediate" Jackson and Bernstein-type inequalities for $\mathscr{R}_{\gamma,\delta}$:

LEMMA 7. *The approximation process $\mathscr{R}_{\gamma,\delta}$, $\gamma, \delta > 0$, satisfies the following inequalities on $L^p(-\infty, \infty)$, $1 \leqslant p \leqslant 2$, $0 < \alpha < \gamma$:*

$$\| R_{\gamma,\delta;\rho}(f) - f\|_{L^p} \leqslant \tilde{C}(\gamma, \delta, \alpha, p)\, \rho^{-\alpha} \|f^{\{\alpha\}}\|_{L^p} \qquad (f \in [L^p]^{\{\alpha\}}), \tag{4.12}$$

$$\| R^{\{\alpha\}}_{\gamma,\delta;\rho}(f)\|_{L^p} \leqslant \tilde{D}(\gamma, \delta, \alpha, p)\, \rho^{\alpha} \|f\|_{L^p} \qquad (f \in L^p(-\infty, \infty)), \tag{4.13}$$

$$\| R_{\gamma,\delta;\rho}(f) - f\|_{L^p} \leqslant C'(\gamma, \delta, \alpha, p)\, \rho^{-\alpha} \|f^{(\alpha)}\|_{L^p} \qquad (f \in [L^p]^{(\alpha)}), \tag{4.14}$$

$$\| R^{(\alpha)}_{\gamma,\delta;\rho}(f)\|_{L^p} \leqslant D'(\gamma, \delta, \alpha, p)\, \rho^{\alpha} \|f\|_{L^p} \qquad (f \in L^p(-\infty, \infty)), \tag{4.15}$$

where $\tilde{C}$, $\tilde{D}$, C', D' are positive constants depending only upon the parameters indicated.

Proof. The first two relations are an immediate consequence of (4.2) and Corollary 1 which interpolates the Jackson and Bernstein-type inequalities of order t^α for $\mathscr{R}_{\gamma,\delta}$ to those of order t^α, $0 < \alpha < \gamma$, with respect to the intermediate space $[L^p]^{\{\alpha\}}$. In view of the fact that $(A')^2 = \tilde{A}^2$, we can derive from Lemma 6, for $0 < \alpha < \gamma$,

$$(X, D((-\tilde{A})^\gamma); \Phi^{(\gamma)}_{\alpha,1}) \subset D((-A')^\alpha) \subset (X, D((-\tilde{A})^\gamma); \Phi^{(\gamma)}_{\alpha,\infty}),$$

using (4.2) for $\gamma = \alpha$ and $\beta = 2r > \alpha$, $r \in \mathbb{N}$, and then (4.4). The application of Corollary 1 yields the inequalities (4.14), (4.15).

We are now able to state

THEOREM 9. *Let $\mathscr{R}_{\gamma,\delta}$ be the linear approximation process given by* (4.11). *Let $\omega(\rho)$ be a positive nonincreasing function on* $[1, \infty)$ *satisfying, for $\rho \to \infty$,*

$$\int_\rho^\infty \omega(u)\, u^{\alpha_1 - 1}\, du = O[\rho^{\alpha_1}\omega(\rho)], \qquad \int_1^\rho \omega^{-1}(u)\, u^{-\alpha_1 - 1}\, du = O[\rho^{-\alpha_1}\omega^{-1}(\rho)],$$

$$\int_1^\rho \omega(u)\, u^{\alpha_2 - 1}\, du = O[\rho^{\alpha_2}\omega(\rho)], \qquad \int_\rho^\infty \omega^{-1}(u)\, u^{-\alpha_2 - 1}\, du = O[\rho^{-\alpha_2}\omega^{-1}(\rho)].$$

If $f \in L^p(-\infty, \infty)$, $1 \leqslant p \leqslant 2$, the following assertions are equivalent for $1 \leqslant q \leqslant \infty$, $0 < \alpha_1 < \alpha_2 \leqslant \rho$ and any positive integer $r > \alpha_2 - \alpha_1$:

(a) $$\int_1^\infty [\omega^{-1}(\rho)\, \| R_{\gamma,\delta;\rho}(f) - f \|_{L^p}]^q \rho^{-1}\, d\rho < \infty,$$

(b) $f \in [L^p]^{\{\alpha_1\}}$, $$\int_1^\infty [\omega^{-1}(\rho)\, \rho^{-\alpha_1} \| R^{\{\alpha_1\}}_{\gamma,\delta;\rho}(f) - f^{\{\alpha_1\}} \|_{L^p}]^q \rho^{-1}\, d\rho < \infty,$$

(c) $$\int_1^\infty [\omega^{-1}(\rho)\, \rho^{-\alpha_2} \| R^{\{\alpha_2\}}_{\gamma,\delta;\rho}(f) \|_{L^p}]^q \rho^{-1}\, d\rho < \infty,$$

(d) $$\int_0^1 [\omega^{-1}(t^{-1})\, K(t^{\alpha_2}, f; L^p, [L^p]^{\{\alpha_2\}})]^q t^{-1}\, dt < \infty,$$

(e) $f^{(\alpha_1)} \in L^p(-\infty, \infty)$,

$$\int_0^1 [\omega^{-1}(t^{-1})\, t^{\alpha_1} \omega_r(t, f)_p]^q t^{-1}\, dt < \infty.$$

Proof. The equivalence of the first four assertions follows by Corollary 2 applied to $Y = D((-A)^{\alpha_2})$ and $Z = D((-A)^{\alpha_1})$ in the previously mentioned fashion, observing the transformation $\rho = t^{-1}$, since Lemma 7 yields all the hypotheses necessary for this application. We can also replace the Riesz-derivative $f^{\{\alpha\}}$ in (b), (c), and (d) by $f^{(\alpha)}$, depending upon which of the inequalities are used in Lemma 7. Now, since $R^{(\alpha_1)}_{\gamma,\delta;\rho}(f) = R_{\gamma,\delta;\rho}(f^{(\alpha_1)})$ for $f \in [L^p]^{(\alpha_1)}$ (this commutativity allows a new proof of the equivalence (a) $\Leftrightarrow$ (b) by reduction theorems, compare (4.7)), Theorem 1 applied to $f^{(\alpha_1)}$ for the case $Y = D(A^r)$, $\Phi = \Phi_{\Omega,q}$, where $\Omega(t) = \omega(t^{-1})\, t_t^{-\alpha_1}$ (compare footnote 3) yields the further equivalent assertion

$$\text{(d)}' \qquad f^{(\alpha_1)} \in L^p(-\infty, \infty),$$
$$\int_0^1 [\omega^{-1}(t^{-1})\, t^{\alpha_1} K(t^r, f; L^p, [L^p]^{(r)})]^q\, t^{-1}\, dt < \infty$$

for any $r > \alpha_2 - \alpha_1$. But this is equivalent to (e) by relation (1.14).

Particular cases of this theorem are to be found in Berens [2], Butzer–Nessel [5], and in the references cited in the latter.

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